

Large- q expansion of the two-dimensional q -state Potts model by the finite lattice method

H. Arisue ^{a *} and K. Tabata ^b

^aOsaka Prefectural College of Technology, 26-12 Saiwai-cho, Neyagawa, Osaka 572, Japan

^bOsaka Institute of Technology, Junior College, Ohmiya, Asahi-ku, Osaka 535, Japan

We have calculated the large- q expansion for the energy and magnetization cumulants at the first order phase transition point in the two-dimensional q -state Potts model to the 21st or 23rd order in $1/\sqrt{q}$ using the finite lattice method. The obtained series allow us to give highly convergent estimates of the cumulants for $q > 4$. The results confirm us the correctness of the conjecture by Bhattacharya *et al.* on the asymptotic behavior of the energy cumulants for $q \rightarrow 4_+$ and a similar new conjecture on the magnetization cumulants.

1. INTRODUCTION

The q -state Potts model[1,2] in two dimensions has been investigated intensively as the test ground for analyzing the phase transition in many physical systems. Especially it is interesting because the order of the phase transition changes when the parameter q is varied, i.e., from the first order for $q > 4$ to the second order for $q \leq 4$. The amplitudes of many quantities at the first order transition point are known exactly, including the latent heat, the spontaneous magnetization[3] and the correlation length[4-6]. Bhattacharya *et al.*[7] made a stimulating conjecture on the asymptotic behavior of the energy cumulants (including the specific heat) at the first order transition point. Bhattacharya *et al.*[8] also made the large- q expansion of the energy cumulants to order 10 in $z \equiv 1/\sqrt{q}$ and analyzing the expansion series with the conjecture they could give the estimates of the cumulants at the transition point for $q \geq 7$ that are better than those given by other methods including the Monte Carlo simulations[9] and the low-(and high-)temperature expansions[10-12]. The series obtained by Bhattacharya *et al.* are, however, not long enough to investigate the behavior of the energy cumulants for q closer to 4.

Here using the finite lattice method[13-15] we

have generated the large- q series for the energy cumulants and the magnetization cumulants at the transition point to order 21 or 23 in z [16]. The finite lattice method can in general give longer series than those obtained by the diagrammatic method especially in lower space (and time) dimensions. In the diagrammatic method, one has to list up all the relevant diagrams and count the number they appear. In the finite lattice method we can skip this job and reduce the main work to the calculation of the partition function for a series of finite size lattices, which can be done without the diagrammatic technique. This method has been used mainly to generate the low- and high-temperature series in statistical systems and the strong coupling series in lattice gauge theory. One of the purposes of our work is to demonstrate that this method is also applicable to the series expansion with respect to the parameter other than the temperature or the coupling constant.

2. ENERGY CUMULANTS

The latent heat \mathcal{L} and the correlation length ξ at the transition point are known to vanish and diverge, respectively, at $q \rightarrow 4_+$ as

$$\mathcal{L} \sim 3\pi x^{-1/2}, \quad (1)$$

$$\xi \sim \frac{1}{8\sqrt{2}}x \quad (2)$$

*This work is supported in part by the Grants-in-Aid of the Ministry of Education (No.08640494)

Table 1

The specific heat for some values of q . The exact correlation length is also listed.

q	C_d	C_o	ξ_d
5	2889(2)	2886(3)	2512.2
6	205.93(3)	205.78(3)	158.9
7	68.738(2)	68.513(2)	48.1
8	36.9335(3)	36.6235(3)	23.9
9	24.58761(8)	24.20344(7)	14.9
10	18.38543(2)	17.93780(2)	10.6
12	12.401336(3)	11.852175(2)	6.5
15	8.6540358(4)	7.9964587(2)	4.2
20	6.13215967(2)	5.36076877(1)	2.7

with $x = \exp(\pi^2/2\theta)$ and $2 \cosh \theta = \sqrt{q}$. Bhattacharya *et al.*'s conjecture says that the n -th energy cumulants $F_{d,o}^{(n)}$ at the first order transition point $\beta = \beta_t$ will diverge at $q \rightarrow 4_+$ as

$$F_d^{(n)}, (-1)^n F_o^{(n)} \sim \alpha B^{n-2} \frac{\Gamma(n - \frac{4}{3})}{\Gamma(\frac{2}{3})} x^{3n/2-2}. \quad (3)$$

This is from the fact that for $\beta \rightarrow \beta_t$ at $q = 4$ (the second order phase transition) the correlation length and the second cumulants diverge as $\xi \sim \lambda|\beta - \beta_t|^{-2/3}$ and $F^{(2)} \sim \mu|\beta - \beta_t|^{-2/3}$, respectively, so that $F^{(n)} \sim \mu \frac{\Gamma(n - \frac{4}{3})}{\Gamma(\frac{2}{3})} (\xi/\lambda)^{3n/2-2}$ and from the assumption that this relation between the correlation length and the energy cumulants are also kept for $q \rightarrow 4_+$ with $\beta = \beta_t$. The constants α and B in Eq.(3) should be common to the ordered and disordered phases from the duality relation for each n -th cumulants.

If this conjecture is true, the product $F^{(n)} \mathcal{L}^{3n-4}$ is a smooth function of θ , so we can expect that the Padé approximation of $F^{(n)} \mathcal{L}^p$ will give convergent result at $p = 3n - 4$. It has been tried for the large- q series obtained by the finite lattice methods for $n = 2, \dots, 6$ both in the ordered and disordered phases, which in fact gives quite convergent Padé approximants for $p = 3n - 4$ and as p leaves from this value the convergence of the approximants becomes bad rapidly. We give in Table 1 the values of the specific heat $C = \beta_t^2 F^{(2)}$ evaluated from these Padé

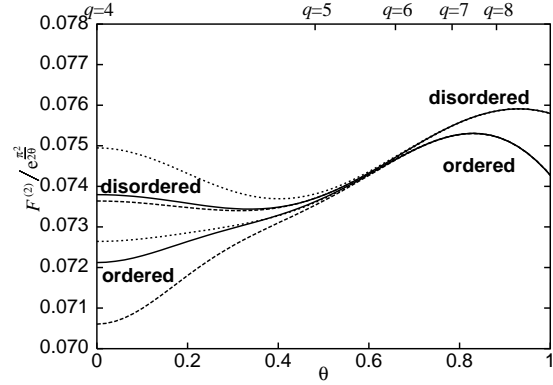


Figure 1. The behavior of the ratio of the second energy cumulants $F_{d,o}^{(2)}$ to x

approximants for some values of q and present in Fig.1 the behavior of the ratio $F^{(2)}/x$ plotted versus θ . These estimates are three or four orders of magnitude more precise than (and consistent with) the previous result for $q \geq 7$ from the large- q expansion to order z^{10} by Bhattacharya *et al.*[8] and the result of the Monte Carlo simulations for $q = 10, 15, 20$ carefully done by Janke and Kappler[9]. What should be emphasized is that we obtained the values of the specific heat in the accuracy of about 0.1 percent at $q = 5$ where the correlation length is as large as 2500. As for the asymptotic behavior of $F^{(n)}$ at $q \rightarrow 4_+$, the Padé data of $F_d^{(2)}/x$ and $F_o^{(2)}/x$ shown in Fig.2 have the errors of a few percent around $q = 4$ and their behaviors are enough to convince us that the conjecture (3) is true for $n = 2$ with $\alpha = 0.073 \pm 0.002$. Furthermore from the conjecture (3) the combination $\{\Gamma(n - \frac{4}{3}) |F^{(n)}| / \Gamma(\frac{2}{3}) F^{(2)}\}^{\frac{1}{n-2}} x^{-\frac{3}{2}}$ is expected to approach the constant B for each $n(\geq 3)$, and in fact the Padé data for every $n(= 3, \dots, 6)$ gives $B = 0.38 \pm 0.05$, which gives strong support to the conjecture also for $n \geq 3$.

3. MAGNETIZATION CUMULANTS

The behavior of the n -th magnetization cumulants $M^{(n)}$ for $\beta \rightarrow \beta_t$ at $q = 4$ is well known as $M_{d,o}^{(n)} \simeq A_{d,o}^{(n)}(\xi)^{\frac{15}{8}n-2}$ and we can make a conjecture parallel to the conjecture for the energy cumulants by Bhattacharya *et al.*: this relation

Table 2

The magnetic susceptibility for some values of q .

q	χ_d	χ_o
5	$9.13(3) \times 10^4$	$9.01(3) \times 10^4$
6	$6.585(4) \times 10^2$	$6.665(4) \times 10^2$
7	70.54(1)	77.31(1)
8	19.359(1)	21.525(1)
9	8.0579(1)	9.0106(2)
10	4.23276(2)	4.73823(4)
12	1.720645(2)	1.918274(3)
15	0.7304214(1)	0.8056969(2)
20	0.309365682(1)	0.33556421(1)

will also be kept in the limit $q \rightarrow 4_+$ with $\beta = \beta_t$, which implies using the asymptotic behavior of the correlation length in Eq.(2) in this limit that

$$M_{d,o}^{(n)} \sim C_{d,o}^{(n)} x^{\frac{15}{8}n-2}. \quad (4)$$

We have tried the Padé approximation of $M^{(n)}\mathcal{L}^p$ for the large- q series generated by the finite lattice method to order z^{21} for $n = 2$ and 3 both in the ordered and disordered phases, which in fact gives quite convergent Padé approximants for $p = 15n/4 - 4$ and as p leaves from this value the convergence of the approximants becomes bad rapidly again. In Table 2 we present the resulting estimates of the magnetic susceptibility $\chi_{d,o} = M_{d,o}^{(2)}$. Our result is much more precise than the Monte Carlo simulation[9] at least by a factor of 100. From the behavior of $M_{d,o}^{(n)}/x^{(15n/8-2)}$ we obtain the coefficients in Eq.(4) as $C_d^{(2)} = 0.0020(2)$, $C_o^{(2)} = 0.0016(1)$ and $C_d^{(3)} = 7.4(5) \times 10^{-5}$, $C_o^{(3)} = 7.9(2) \times 10^{-5}$. These convince us that the conjecture made for the magnetization cumulants is also true.

4. SUMMARY

The large- q series for the energy and magnetization cumulants generated by the finite lattice method in the two-dimensional q -state Potts model give very precise estimates of the cumulants for $q > 4$ and they confirm the correctness of the conjecture that the relation between the

cumulants and the correlation length for $q = 4$ and $\beta \rightarrow \beta_t$ (the second order phase transition) is kept in their asymptotic behavior for $q \rightarrow 4_+$ at $\beta = \beta_t$ (the first order transition point). If this kind of relation is satisfied for the quantities at the first order phase transition point in more general systems as the asymptotic behavior when the parameter of the system is varied to make the system close to the second order phase transition point, it would serve as a good guide in investigating the property of the system.

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